

CYCLIC EVOLUTION ON GRASSMANN MANIFOLD AND BERRY PHASE

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ABSTRACT. For a given k -dimensional subspace V_0 in a Hilbert space \mathcal{H} and a unitary transformation $g_0 : V_0 \longrightarrow V_0$, we find a path in the Grassmann manifold the monodromy of which coincides with g_0 .

Let \mathcal{H} be a finite-dimensional Hilbert space; $U(\mathcal{H})$ be the Lie group of unitary transformations of \mathcal{H} and $\mathfrak{u}(\mathcal{H})$ be the corresponding Lie algebra. For any positive integer k , the Grassmann manifold $\text{Gr}_k(\mathcal{H})$ is defined as the set of all k -dimensional subspaces of \mathcal{H} . This manifold can also be described as the set of corresponding orthogonal projectors

$$\text{Gr}_k(\mathcal{H}) = \left\{ P : \mathcal{H} \longrightarrow \mathcal{H} \mid P \text{ is linear, } P^\dagger = P, \text{ tr}(P) = k \right\}.$$

As it is well-known for any given Hamiltonian $H \in \mathfrak{u}(\mathcal{H})$ the corresponding Schrödinger equation is defined as the equation of the form

$$(1) \quad \dot{\psi}(t) = H(\psi(t)), \quad \psi(t) \in \mathcal{H}, \quad t \in \mathbb{R}, \quad \psi(0) = \psi_0,$$

and

$$\Phi_H = \{ \exp(tH) \mid t \in \mathbb{R} \}$$

is the corresponding one-parameter family of unitary transformations of \mathcal{H}

Obviously, the equation (1) defines a dynamical system on the Grassmann manifold $\text{Gr}_k(\mathcal{H})$:

$$(2) \quad \dot{P}(t) = [H, P(t)], \quad t \in \mathbb{R}$$

and the corresponding one-parameter group of diffeomorphisms of $\text{Gr}_k(\mathcal{H})$ is defined by the action of the group Φ_H on $\text{Gr}_k(\mathcal{H})$. The action of the group Φ_H for the projector representation of $\text{Gr}_k(\mathcal{H})$, is

$$P \mapsto \exp(tH)P \exp(-tH).$$

For a given k -dimensional subspace $V_0 \in \mathcal{H}$, we are interested in Hamiltonians $H \in \mathfrak{u}(\mathcal{H})$ such that, after the time period $t = 1$, the one-parameter group Φ_H brings V_0 to itself. In other words, for a given point $P_0 \in \text{Gr}_k(\mathcal{H})$ we are looking for Hamiltonians $H \in \mathfrak{u}(\mathcal{H})$ such that the trajectory of the equation (2) through the point P_0 is closed:

$$\exp(H)P_0 \exp(-H) = P_0.$$

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When the transformation $\exp(H)$ brings the subspace V_0 to itself, it defines a unitary transformation

$$g_0 = \exp(H)|_{V_0} : V_0 \longrightarrow V_0.$$

Remark 1. *In fact, the unitary transformation $g_0 : V_0 \longrightarrow V_0$, induced by the one-parameter flow $\{\exp(tH) \mid t \in \mathbb{R}\}$, is the well-known Berry phase and can be decomposed in so called “dynamical” and “geometrical” factors. Here we don’t concern this decomposition and consider the Berry phase as a “single whole”.*

After this, we can reformulate our problem as

Problem 1. *for a given k -dimensional subspace $V_0 \in \mathcal{H}$ and a unitary transformation $g_0 : V_0 \longrightarrow V_0$, find a skew-hermitian operator $H : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\exp(H)V_0 = V_0$ and $\exp(H)|_{V_0} = g_0$.*

It is clear that when $[H, P_0] = 0$, the solution of the Schrödinger equation (2) with the initial condition $P(0) = P_0$ is constant: $P(t) = P_0$, $t \in [0, 1]$, therefore, it is preferable that the operator H be such that $[H, P_0] \neq 0$.

Remark 2. *In [1] it is considered the similar problem, but for the “geometric” factor of the Berry phase corresponding to the cyclic trajectory on the Grassmannian $Gr_k(\mathcal{H})$ defined by $\exp(tH)$, $t \in [0, 1]$.*

Further we will discuss the solution of Problem 1.

Let $m = \dim(V_0)$ and $\mathcal{E}_0 = \{e_0, \dots, e_m\}$ be an orthonormal basis of V_0 consisting of eigenvectors of the operator g_0 :

$$g_0(e_k) = u_k \cdot e_k, \quad u_k \in \mathbb{C}, \quad |u_k| = 1, \quad k = 1, \dots, m.$$

Consider an orthonormal extension of the basis \mathcal{E}_0 to the basis of the entire Hilbert space \mathcal{H} :

$$\mathcal{E} = \mathcal{E}_0 \bigcup \mathcal{E}_1, \quad \mathcal{E}_1 = \{e_{m+1}, \dots, e_n\} \subset V_0^\perp,$$

where $n = \dim(\mathcal{H})$, and define the unitary operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ as

$$g|_{V_0} = g_0, \quad g(e_{m+1}) = u_m \cdot e_{m+1} \quad \text{and} \quad g(e_p) = e_p \quad \text{for} \quad m+2 \leq p \leq n.$$

In other words, we set that the vectors $e_1, \dots, e_m, e_{m+1}, \dots, e_n$ are eigenvectors of g , the restriction of the operator g to the subspace V_0 coincides with g_0 , the eigenvalues of g on e_m and e_{m+1} are equal and its eigenvalues on the vectors e_{m+2}, \dots, e_n are equal to 1. The matrix of the operator g in

the basis \mathcal{E} is of the form

$$U = \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \\ & \ddots & u_m & 0 \\ \vdots & & 0 & u_m & \ddots \\ & & & \ddots & 1 \\ & & & & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \end{bmatrix},$$

and the matrix of the projector P_0 in the same basis is

$$A = \begin{pmatrix} \mathbf{1}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-m} \end{pmatrix},$$

where $\mathbf{1}_m$ denotes $m \times m$ identity matrix and $\mathbf{0}_{n-m}$ denotes $(n-m) \times (n-m)$ zero matrix. Hence, the problem is reduced to the finding a matrix H such that $\exp(H) = U$ and $[H, A] \neq 0$.

Assume $u_1 = e^{i\lambda_1}, \dots, u_m = e^{i\lambda_m}$, $\lambda_k \in \mathbb{R}$, $k = 1, \dots, m$. Obviously, the number u_m can also be written as $u_m = e^{i(\lambda_m + 2\pi n)}$, $n \in \mathbb{Z}$. For any unitary transformation $\omega \in U(2)$ let $H \equiv H_\omega$ be the following block-diagonal matrix

$$H_\omega = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & \Omega & 0 \\ 0 & 0 & \mathbf{0}_{n-m-1} \end{pmatrix},$$

where H_1 is the $(m-1) \times (m-1)$ diagonal matrix: $H_1 = \text{diag}[i\lambda_1, \dots, i\lambda_{m-1}]$; and Ω is the matrix

$$\Omega = \omega \begin{pmatrix} i\lambda_m & 0 \\ 0 & i(\lambda_m + 2\pi n) \end{pmatrix} \omega^{-1}.$$

It is clear that $\exp(H_\omega)$ is

$$\exp(H_\omega) = \begin{pmatrix} \exp(H_1) & 0 & 0 \\ 0 & \exp(\Omega) & 0 \\ 0 & 0 & \mathbf{1}_{n-m-1} \end{pmatrix}.$$

Since

$$\exp(\Omega) = \omega \exp \begin{pmatrix} i\lambda_m & 0 \\ 0 & i(\lambda_m + 2\pi n) \end{pmatrix} \omega^{-1} = \begin{pmatrix} u_m & 0 \\ 0 & u_m \end{pmatrix},$$

we obtain $\exp(H_\omega) = U$. On the other hand, it is clear that $[H_\omega, A] = 0$ if and only if $[\epsilon, \Omega] = 0$, where

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and the latter happens only when Ω is diagonal.

To summarize, we can say that we have a family of solutions of Problem 1 depending on the unitary matrix $\omega \in U(2)$ and the integer n .

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